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Toward a New Engineering Theory of Bending: Fundamentals

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The classical engineering theory of bending due to Bernoulli and Euler serves as a cornerstone for structural analysis and design. Limitations of this theory, however, become apparent in flexural wave propagation studies; it predicts infinite phase velocity as the wavelength becomes shorter. This theoretical deficiency is corrected by Timoshenko theory which accounts for transverse shear deformation. A thorough study of exact elasticity solutions reveals that there are two additional effects that are the same order as transverse shear in bending behavior. A new theory accounting for them is presented, along with several applications. The new equations are no more complicated than those of Timoshenko-type theory, yet they yield solutions which are exact or indistinguishable from exact in the examples studied.

Introduction

THE classical engineering theory of bending due to Bernoulli and Euler dates back to 1705 and precedes the theory of elasticity by over 100 years.[1] It has long been recognized as a convenient approximation for slender beams and serves as a cornerstone for structural analysis and design. Extensions due to Kirchhoff and Love expand the scope of this first approximation theory to plates and shells.

Limitations of Bernoulli-Euler theory become apparent in studying the propagation of elastic flexural waves of short wavelength. It predicts infinite phase velocity for harmonic waves as the wavelength becomes shorter. This result is, of course, physically absurd. This theoretical deficiency is corrected by the theory proposed by Timoshenko.[2] In Timoshenko theory, the influence of transverse shear deformations are accounted for, which results in a finite limit for phase velocity.

There have been a number of attempts to improve upon Timoshenko theory. Most of them are founded on the original Timoshenko assumptions,[2] which are that cross sections remain plane after bending and that shear of cross sections relative to one another is permitted to occur. The differences in various proposed equations are due primarily to the selection of the transverse shear contribution to the response according to different ad hoc criteria. An historical sketch of the development of engineering bending theory and a discussion of attempts at refinement appear in Ref. 3. The emergence of laminated, fiber reinforced advanced composite materials for structural applications and their unusual properties provides the motivation for the present work.

Use of fiber reinforced resin matrix composite materials in aerospace vehicles is increasing. This is primarily due to their superior mechanical properties and the ease with which they can be tailored to a specific application. The directional nature of their properties, however, poses unique challenges for the analyst. Consider, for example, a single layer or lamina made of a composite material. The extensional modulus along the direction of fibers is usually very large relative to the extensional moduli in the lateral directions and the shear moduli. This is a marked departure from conventional isotropic materials. Consequently, the relative importance of physical effects is influenced by the directional nature of properties and their relative magnitude. Transverse shear deformations, for example, are much more pronounced for composite structures.

This work has two primary objectives. The first is to delineate some features of bending behavior in a way which is new and informative. An analysis of bending behavior is described which utilizes an exact solution from the theory of elasticity for isotropic materials. A unique feature is the use of tracer constants in order to track the contributions due to various physical effects throughout the course of the analysis. With the aid of insight from this analysis, the second objective, the basis of a new engineering theory, is achieved. The new theory is applied to elementary static applications for beam-type structures which illustrate its use and permit comparisons with exact solutions to establish its validity.

The scope of this work is restricted to static, planar bending situations. In its present form, the theory applies to beams with thin rectangular cross sections which respond to planar bending in plane stress or to infinitely wide plates which respond in plane strain (cylindrical bending). All equations will be written for plane stress and treated as exact, although plane stress is an approximate state that is valid for thin cross sections.[4] The planar bending restriction is not intrinsic to the subjects being considered. This context is simply a basic step to take prior to confronting the additional complexities of fully three-dimensional behavior. Both isotropic and orthotropic materials are considered. Beams of orthotropic material are the simplest type of structures where composite material behavior can be studied.

Preliminary Analysis

Introductory Remarks

As a first, important step in an analysis of bending behavior, a plane stress elasticity solution for a simply supported beam under uniformly distributed loading is studied. It is possible to identify the individual contributions due to various factors affecting beam response. An assessment of their relative importance, therefore, can be made.

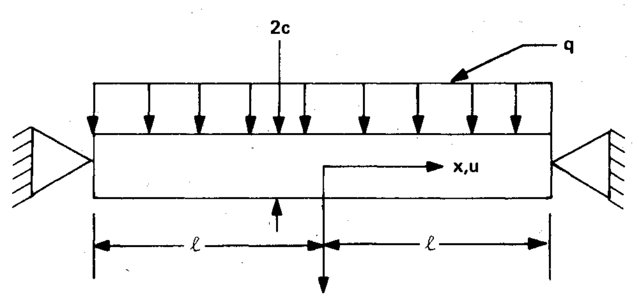

Fig. 1 Uniformly loaded simply supported beam and coordinate system.

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Problem Definition and Solution

The two-dimensional elasticity solution for a simply supported beam under uniformly distributed loading is given in the text by Timoshenko and Goodier.[4] It is valid for very thin rectangular beams in the plane stress form. For infinitely wide plates, the same solution remains valid if a transformation of elastic constants for plane strain is employed.

The beam and coordinate system are shown in Fig. 1. The length of the beam is 2ℓ and the depth is $2c$. The width of the beam is taken as unity for convenience. The beam is bent by a uniformly distributed load of intensity q applied to its upper surface. The midspan of the beam centroidal axis is chosen as the origin for the coordinate axes x and z. $z=+c$ and $z=-c$ correspond to the bottom and top surfaces of the beam. The notation and convention are shown.

For the stresses, the usual convention and notation are followed: σ_{xx} is the axial stress, σ_{zz} the transverse normal stress, and σ_{xz} the transverse shear stress. They are given by the following expressions.

$$\sigma_{xx}=\frac{q}{2I}(\ell^2-x^2)z+\underline{\frac{q}{2I}\left(\frac{2z^3}{3}-\frac{2c^2z}{5}\right)} \tag{1}$$

$$\sigma_{zz}=-\frac{q}{2I}\left(\frac{z^3}{3}-c^2z+\frac{2c^3}{3}\right) \tag{2}$$

$$\sigma_{xz}=-\frac{q}{2I}(c^2-z^2)x \tag{3}$$

These stresses satisfy all the governing differential equations and the stress boundary conditions on the upper and lower surfaces. On the ends $x=\pm\ell$, the stress boundary conditions are satisfied in an overall Saint Venant sense. I is the second moment of the cross-sectional area and is $2c^3/3$ for the rectangular section under consideration.

As an aid in this analysis, three tracer constants, α_a, α_n, and α_s, are introduced. They are defined and used so as to facilitate keeping track of three distinct contributions to the response. The first term of Eq. (1) corresponds to the bending stress given by classical Bernoulli-Euler theory. The underlined term is a stress contribution which will be called the "nonclassical axial stress." It produces no resultant force or moment and is, therefore, a self-equilibrating stress. α_a is the tracer constant associated with this contribution. If $\alpha_a=1$, this contribution is fully accounted for. If $\alpha_a=0$ in the following, however, it is ignored and the Bernoulli-Euler axial stress distribution is recovered. For example, the axial stress is written using this convention in the form

$$\sigma_{xx}=\frac{q}{2I}(\ell^2-x^2)z+\alpha_a\frac{q}{2I}\left(\frac{2}{3}z^3-\frac{2}{5}c^2z\right) \tag{1a}$$

α_n and α_s are defined analogously and are associated with contributions due to σ_{zz} and σ_{xz}, respectively.

The displacement components u and w are shown in Fig. 1. Expressions for them can be obtained by using Hooke's law and the strain displacement relations. The following boundary conditions at the ends $x=+\ell$ and $x=-\ell$ are imposed:

$$w(\ell,0)=w(-\ell,0)=0 \tag{4}$$

They represent support conditions applied at the beam axis. Also, from the symmetry requirement

$$u(0,0)=0 \tag{5}$$

These conditions are sufficient to prevent rigid body motion.

The expressions for u and w are

$$u(x,z)=\frac{q}{2EI}\left[\left(\ell^2x-\frac{x^3}{3}\right)z+\alpha_a\left(\frac{2}{3}z^3-\frac{2c^2z}{5}\right)x+\alpha_n\nu\left(\frac{z^3}{3}-c^2z+\frac{2c^3}{3}\right)x\right] \tag{6}$$

$$w(x,z)=w(x,0)-\frac{q}{2EI}\left[\left(\frac{z^4}{12}-\frac{c^2z^2}{2}+\frac{2c^3z}{3}\right)\alpha_n+\nu(\ell^2-x^2)\frac{z^2}{2}+\nu\alpha_a\left(\frac{z^4}{6}-\frac{c^2z^2}{5}\right)\right] \tag{7}$$

In Eq. (7), $w(x,0)$ is the vertical deflection of the beam centroidal axis due to bending. It is given by the equation

$$w(x,0)=\delta-\frac{q}{2EI}\left[\frac{\ell^2x^2}{2}-\frac{x^4}{12}+\left[(1+\nu)\alpha_s-\left(\frac{\alpha_a}{5}+\frac{\alpha_n\nu}{2}\right)\right]c^2x^2\right] \tag{8}$$

where

$$\delta=\frac{5q\ell^4}{24EI}\left[1+\frac{12c^2}{5\ell^2}\left[(1+\nu)\alpha_s-\frac{\alpha_a}{5}-\frac{\alpha_n\nu}{2}\right]\right] \tag{9}$$

δ is the deflection at the midspan of the beam.

Analysis of Beam Response

The major differences between the classical Bernoulli-Euler theory and the elasticity solution can be clearly identified in Eq. (9). The bracketed term represents the correction to the former due to the presence of contributions identified by the tracer constants α_s, α_a, and α_n. Note that the contributions due to all three effects—transverse shear, nonclassical axial stress, and transverse normal strain—are of the same order of magnitude. A static version of Timoshenko's theory[2] includes only the terms associated with α_s.

The corrections shown in Eq. (9) were known to earlier authors.[5,6] However, they did not differentiate among the various contributions. This differentiation provides the key ingredient for the establishment of a rational engineering theory. Goodier[7] suspected that the other influences beside transverse shear were important, but offered no means of estimating them quantitatively and no concrete examples of their contribution to beam response. The approach adopted here makes the matter transparent and settles the issue for this example.

Conclusions

On the basis of the foregoing analysis, the following conclusions are reached.

1) A Timoshenko-type transverse shear theory does not contain the necessary physical ingredients to treat problems with distributed loadings.

2) Transverse shear, nonclassical axial stress, and transverse normal strain make contributions to the response that are of the same order of magnitude. A theory that is purported to be more accurate or complete than classical theory must, therefore, correctly account for all of these influences.

Foundations of a New Theory

Objectives

An engineering theory is one in which assumptions or approximations are introduced in order to simplify the governing equations or facilitate their solution. Hopefully,

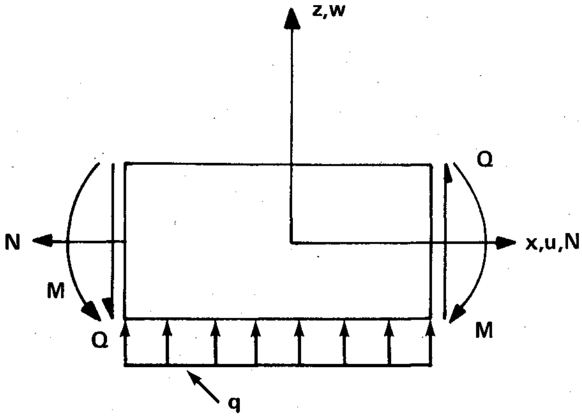

Fig. 2 Sign convention.

only a little accuracy is sacrificed for a considerable reduction in computational labor. The intent is to encompass the heart of the problem under consideration. The primary objective of this work is the development of a foundation for such a theory for bending that is consistent, reliable, and simple to use. Furthermore, it must account for the three effects that were clearly identified previously: transverse shear strain, nonclassical axial stress, and transverse normal strain.

The standard of comparison for results that is used herein is the plane stress solution to the equations of elasticity theory for the problem in question.

A second objective is to obtain stress estimates that are improvements over those provided by classical bending theory. This must be accomplished if the influence of nonclassical axial stress is to be properly accounted for.

Statically Equivalent Stresses

Equilibrium of a beam element is governed by overall equations containing resultant axial force, shear force, and bending moment. The sign convention and notation for these appear in Fig. 2. The equilibrium equations are

$$N_{,x}=0 \tag{10}$$

$$Q_{,x}+q=0 \tag{11}$$

$$M_{,x}-Q=0 \tag{12}$$

The force and moment resultants are defined in terms of stresses as

$$N=\int_{-c}^{c}\sigma_{xx}dz \tag{13}$$

$$Q=\int_{-c}^{c}\sigma_{xz}dz \tag{14}$$

$$M=\int_{-c}^{c}\sigma_{xx}zdz \tag{15}$$

In the preceding, a rectangular cross section of unit width is assumed as before.

According to classical theory, the stresses are

$$\sigma_{xx}=(N/A)+(Mz/I) \tag{16}$$

$$\sigma_{xz}=(Q/2I)(c^2-z^2) \tag{17}$$

$$\sigma_{zz}=\frac{Q_{,x}}{2I}\left(\frac{z^3}{3}-c^2z+\frac{2}{3}c^3\right) \tag{18}$$

A is the cross-sectional area, which is $2c$ for the rectangular cross section under consideration. These stresses are statically equivalent to the applied loads and satisfy the stress equations of equilibrium. In addition, Eqs. (13-15) are satisfied, as are appropriate stress conditions at $z=c$ and $z=-c$.

The stresses, just given, although not exact, serve as a first approximation. They will be used subsequently to develop approximations for the displacement components.

Kinematics

Classical Bernoulli-Euler theory is based upon a kinematic assumption that is equivalent to ignoring, and hence setting to zero, transverse normal strain and transverse shear strain. Timoshenko-type shear deformation theories account for transverse shear strain but still do not permit transverse normal strain. On the basis of the previous analysis of the simply supported beam example, it appears necessary to completely abandon the Bernoulli-Euler kinematic assumption. In order to obtain some simplification from the complete elasticity equations, however, an assumption that facilitates the analysis is required.

The central assumption that replaces the Bernoulli-Euler hypothesis in the present development is that the statically equivalent stresses in Eqs. (16-18) can be used to estimate the transverse normal strain and transverse shear strain. Note that this is an assumption regarding stresses. It is not a kinematic assumption. This is in sharp contrast to classical and Timoshenko-type shear deformation theories.

The development will be carried out for orthotropic materials with principle material directions corresponding to axes of the beam. The appropriate form of Hooke's law for plane stress (beams of thin rectangular cross section) is

$$\epsilon_{xx}=(1/E_{11})(\sigma_{xx}-\nu_{13}\sigma_{zz}) \tag{19}$$

$$\epsilon_{zz}=\frac{\sigma_{zz}}{E_{33}}-\frac{\nu_{13}}{E_{11}}\sigma_{xx} \tag{20}$$

$$\gamma_{xz}=\sigma_{xz}/G_{13} \tag{21}$$

ϵ_{xx}, ϵ_{zz}, γ_{xz} are the axial strain, transverse normal strain, and transverse shear strain, respectively. E_{11} and E_{33} are elastic moduli associated with the x and z directions. ν_{13} is Poisson's ratio and G_{13} the transverse shear modulus. In addition, the strain components are related to the displacement components by the following strain displacement relations.

$$u_{,x}=\epsilon_{xx};\qquad w_{,z}=\epsilon_{zz};\qquad u_{,z}+w_{,x}=\gamma_{xz} \tag{22}$$

Equations (16), (18), (20), and (22) permit the transverse normal strain to be approximated as

$$w_{,z}\cong-\frac{\nu_{13}}{E_{11}}\left(\frac{N}{A}+\frac{Mz}{I}\right)+\underline{\frac{Q_{,x}}{2E_{33}I}\left(\frac{z^3}{3}-c^2z+\frac{2}{3}c^3\right)} \tag{23}$$

Integration of this equation results in the following expression for the lateral displacement component w:

$$w=W(x)-\frac{\nu_{13}}{E_{11}}\left(\frac{Nz}{A}+\frac{Mz^2}{2I}\right)+\underline{\frac{Q_{,x}}{2E_{33}I}\left(\frac{z^4}{12}-\frac{c^2z^2}{2}+\frac{2}{3}c^3z\right)} \tag{24}$$

$W(x)$ is the lateral deflection of the beam axis ($z=0$), which is an unknown function to be determined.

The axial component of displacement u can be estimated as follows. Equations (17), (21), and (22) permit $u_{,z}$ to be expressed as

$$u_{,z}=\sigma_{xz}/G_{13}-w_{,x}=\frac{Q}{2G_{13}I}(c^2-z^2)-W_{,x}+\frac{\nu_{13}}{2E_{11}I}Qz^2$$

$$-\frac{Q_{,xx}}{2E_{33}I}\left(\frac{z^4}{12}-\frac{c^2z^2}{2}+\frac{2}{3}c^3z\right) \tag{25}$$

This expression is integrated to yield

$$u=U(x)-zW_{,x}+\frac{\nu_{13}}{E_{11}I}Q\frac{z^3}{6}+\frac{Q}{2G_{13}I}\left(c^2z-\frac{z^3}{3}\right)$$

$$\underline{-\frac{Q_{,xx}}{2E_{33}I}\left(\frac{z^5}{60}-\frac{c^2z^3}{6}+\frac{1}{3}c^3z^2\right)} \tag{26}$$

$U(x)$ is the axial deflection of the beam axis, which is an unknown function to be determined.

The static displacement field is completely described by Eqs. (24) and (26). This approximate displacement field was determined by estimating precisely the strains that are ignored in classical theory. U and W, the axis displacement components, emerge as natural kinematic variables. If $\nu_{13}\to 0$ and $E_{33}\to\infty$ in Eqs. (24) and (26), a transverse shear theory is obtained that includes the effects of cross section warping. If, in addition, $G_{13}\to\infty$, then the classical Bernoulli-Euler kinematic assumption is recovered.

Considerable simplification is achieved if the underlined terms in Eqs. (24) and (26) are neglected. These terms are associated with higher derivatives of the shear force Q than the remaining terms. This simplification is adopted here. Its full implication will be discussed in a future paper. The accuracy of this approximation is related to how rapidly the applied load q varies with x.

Refined Axial Stress Distribution

The axial stress σ_{xx} is the largest and most important stress component. An accurate knowledge of it is often all that is needed in a practical application. A refined estimate that improves Eq. (16) is central, therefore, to the improvements that are sought.

Equations (18), (19), and (26) can be utilized to produce a refined axial stress expression.

$$\sigma_{xx}=E_{11}u_{,x}+\nu_{13}\sigma_{zz}=E_{11}\left[U_{,x}-zW_{,xx}+\frac{\nu_{13}}{E_{11}I}Q_{,x}\frac{z^3}{6}\right.$$

$$\left.+\frac{Q_{,x}}{2G_{13}I}\left(c^2z-\frac{z^3}{3}\right)\right]+\nu_{13}\frac{Q_{,x}}{2I}\left(\frac{z^3}{3}-c^2z+\frac{2}{3}c^3\right) \tag{27}$$

In the preceding, contributions due to the underlined terms in Eq. (26) are not included. Notice that the stress distribution throughout the thickness is not linear as in the classical approximation Eq. (16).

Relationships for the axial force and bending moment are obtained by using Eqs. (13), (15), and (27). The results are

$$N=\left(E_{11}U_{,x}+\frac{\nu_{13}c^3}{3I}Q_{,x}\right)A \tag{28}$$

$$M=-E_{11}IW_{,xx}+\left(\frac{4}{5}k_x+\frac{\nu_{13}}{2}\right)c^2Q_{,x} \tag{29}$$

The parameter k_x is $(E_{11}/2G_{13}-\nu_{13})$; it is unity for an isotropic material. Equations (28) and (29) permit Eq. (27) to be rewritten as

$$\sigma_{xx}=\frac{N}{A}+\frac{Mz}{I}+\underline{\frac{Q_{,x}}{3I}k_x\left(\frac{3}{5}c^2z-z^3\right)} \tag{27a}$$

The underlined term is the nonclassical axial stress contribution, which is the desired refinement.

Summary

The governing equations for the new theory can be summarized now. They encompass four categories. Overall beam-type equations consist of the equilibrium Eqs. (10-12) and the constitutive Eqs. (28) and (29). In addition, two sets of equations provide the distributions of stresses and displacements throughout the structure. The first set for stresses consists of Eqs. (27a), (17), and (18). The second for displacements is composed of Eqs. (24) and (26) with the underlined terms omitted.

The preceding collection of equations requires the specification of boundary conditions. A virtual work development based upon U and W as displacements to be varied yields the classical boundary conditions as options. Thus, N or U, Q or W, and M or $W_{,x}$ must be specified at the ends of the beam.

It is often convenient to introduce an intermediate kinematic variable that is related to rotation of the beam cross section. Three commonly used variables are considered next. The first is the rotation of the cross section at the beam axis ϕ_1.

$$\phi_1=u_{,z}(x,0)=\frac{Qc^2}{2G_{13}I}-W_{,x} \tag{30}$$

Another is the rotation-related variable ϕ_2, which is defined by the following equation:

$$\phi_2=\frac{1}{I}\int_{-c}^{c}uz\,dz=-W_{,x}+\frac{3}{10A}\left(\frac{\nu_{13}}{E_{11}}+\frac{4}{G_{13}}\right)Q \tag{31}$$

This variable naturally arises in Reissner's development of plate bending theory[8] based upon the complementary energy principle.

The third is the mean rotation of the cross section ϕ_3.

$$\phi_3=\frac{1}{2c}\int_{-c}^{c}u_{,z}dz=\frac{1}{2c}[u(x,c)-u(x,-c)]$$

$$=\frac{Qc^2}{3I}\left(\frac{1}{G_{13}}+\frac{\nu_{13}}{2E_{11}}\right)-W_{,x} \tag{32}$$

In a Timoshenko-type theory, since u is linear in z, all of the preceding definitions are equivalent. Equation (30) is the actual definition used in the original paper.[2] These variables permit different models for simulating clamped end conditions to be defined.

Discussion

The development of the equations requires no ad hoc kinematic assumptions or use of a variational principle. The central assumption is that the transverse normal and shear strain components can be estimated from classical stresses. The equations have the following properties:

1) Stress and displacement distributions throughout the structure are found in terms of the response variables associated with the axis.

2) Nonclassical axial stress and cross section warping effects, transverse shear strain and transverse normal strain are all accounted for in a rational manner.
3) The equations can be shown to yield exact results for the case of uniformly distributed lateral loading.
4) For nonuniform loading, some of the equations are approximate: the stresses are not exactly in equilibrium and the stresses and displacements are not exactly compatible.
5) The equations are as simple to apply as static Timoshenko-type shear deformation theories.

Items 1-3 and 5 are strong points in favor of the new equations. Item 4 imposes some limitations on the validity of the theory, which will be thoroughly discussed in a future paper, but it is responsible for the simplicity that is achieved. The level of stress approximation which results is analogous to that suggested by Seewald[6] for isotropic materials.

In the process of solving a particular bending problem, the only apparent difference from application of a static Timoshenko-type shear deformation theory is the value for the coefficient of the $Q_{,x}$ term in Eq. (29). As the applications will demonstrate, this seemingly minor difference, together with the use of Eqs. (24), (26), and (27a), produces substantially improved results.

Applications

Introductory Remarks

In order to illustrate the theory, three elementary applications for uniform beams subjected to uniformly distributed loading applied to the upper surface, analogous to the situation shown in Fig. 1, will be presented. One special case of a linearly varying load is studied to illustrate a particular point. Comparisons are made with the exact solution, classical Bernoulli-Euler theory and the original static Timoshenko theory[2] in each case.

Solutions are derived for orthotropic beams, and corresponding results for isotropic beams are obtained by specialization. Poisson's ratio is taken to be 0.3 throughout. For orthotropic beams, E_{11}/G_{13} is taken to be 30; this is a typical value for a modern graphite/epoxy composite material.

In presenting results, appropriate response variables are nondimensionalized with respect to the corresponding values obtained from Bernoulli-Euler theory. This practice permits easy recognition of departures from classical theory predictions.

Response can be separated into bending and stretching. Stretching is governed by Eqs. (10) and (28), bending by Eqs. (11), (12), and (29). The bending problem involving M, Q, and W must be solved first. N and U, stretching variables, are determined secondarily. For the present purposes, only the bending portion of the response is discussed.

Simply Supported Beam

The exact solution for a simply supported (SS) isotropic beam was presented earlier. The precise boundary conditions that have been imposed at the ends are

$$\text{SS:} \quad M=0, \quad W=0 \tag{33}$$

Such a beam is shown in Fig. 1. It is a statically determinate structure, so the moment and shear distributions are known.

The response of the beam is defined if the axis lateral deflection $W(x)$ is found. For this type of end restraint, W can be expressed as

$$W=\delta-\frac{q}{4E_{11}I}\left[\left(\ell^2x^2-\frac{x^4}{6}\right)+\frac{K}{2}H^2x^2\right] \tag{34}$$

In Eq. (34) the constant K is

$$K=\tfrac{1}{2}\left(\alpha_s\frac{E_{11}}{G_{13}}-\alpha_n\nu_{13}\right)-\alpha_a\frac{k_x}{5} \tag{35}$$

The tracer constants introduced earlier are utilized to identify the origin of the various contributions to K. δ is the maximum or midspan deflection, which is

$$\delta=\frac{5q\ell^4}{24E_{11}I}\left[1+\frac{12}{5}K\left(\frac{H}{L}\right)^2\right] \tag{36}$$

$L=2\ell$ is the total length of the beam and $H=2c$ the depth of the cross section.

The solution by the present theory corresponds to $\alpha_a=\alpha_n=\alpha_s=1$ in Eq. (35); it is exact for this problem. If K is set to zero in Eqs. (34) and (36), then the Bernoulli-Euler result is obtained. If $\alpha_a=\alpha_n=0$ and $\alpha_s=1$ in Eq. (35), the static Timoshenko theory prediction is recovered. Timoshenko theory overestimates the deflection (underestimates stiffness) for this case.

Cantilever Beam

For a cantilever beam, it is convenient to take the origin of coordinates, $x=0$, at the free end. $x=L$ corresponds to the clamped end. Unlike the more elementary theories, the present theory does not suggest a unique, simple model for a clamped or fixed end. Three rotation variables were introduced earlier in Eqs. (30-32). Three clamping models, based upon these variables, will be discussed.

All results can be cast in a common format. The three clamping models are denoted C1, C2, and C3. They correspond to the following conditions:

$$\text{C1:} \quad W=0, \qquad \phi_1=0 \tag{37}$$

$$\text{C2:} \quad W=0, \qquad \phi_2=0 \tag{38}$$

$$\text{C3:} \quad W=0, \qquad \phi_3=0 \tag{39}$$

These models do not eliminate cross section warping, but they permit solutions by elementary means. The cantilever beam is statically determinate with M and Q vanishing at the free end. At the fixed end, one of the definitions of clamping just given must be imposed. The rotation variables can be expressed in the common form

$$\phi_i=K_i\frac{Qc^2}{E_{11}I}-W_{,x}; \qquad i=1,2,3 \tag{40}$$

The constants K_1-K_3 are

$$K_1=\frac{\alpha_s}{2}\frac{E_{11}}{G_{13}} \tag{41}$$

$$K_2=\frac{1}{2}\left[\alpha_s\frac{E_{11}}{G_{13}}-\frac{\alpha_a}{5}\left(\frac{E_{11}}{G_{13}}-\nu_{13}\right)\right] \tag{42}$$

$$K_3=\frac{1}{2}\left[\alpha_s\frac{E_{11}}{G_{13}}-\frac{\alpha_a}{3}\left(\frac{E_{11}}{G_{13}}-\nu_{13}\right)\right] \tag{43}$$

The lateral deflection can be conveniently expressed as follows:

$$W=\delta_i-\frac{q}{E_{11}I}\left[\left(\frac{L^3}{6}+(K_i-K)c^2L\right)x+\frac{KH^2}{8}x^2-\frac{x^4}{24}\right]; \qquad i=1,2,3 \tag{44}$$

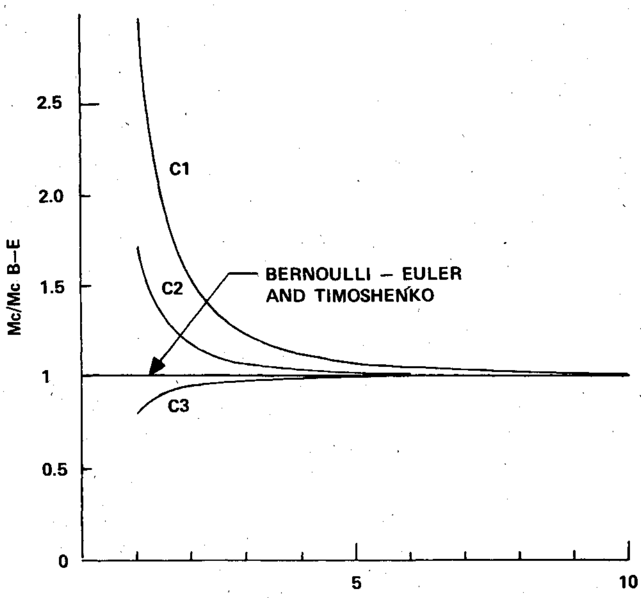

Fig. 3 Central moment ratio for a clamped isotropic beam.

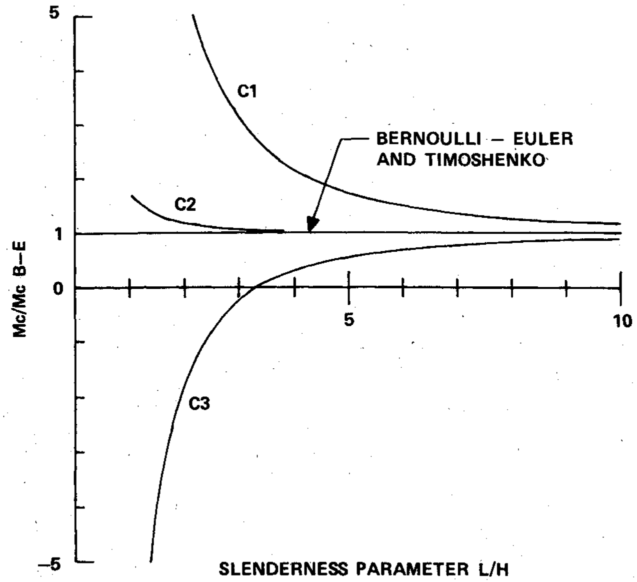

Fig. 4 Central moment ratio for a clamped orthotropic beam.

The constants δ_i ($i=1,2,3$) are the beam tip deflections; they are found from

$$\delta_i = \frac{qL^4}{8E_{11}I}\left[1+(2K_i-K)\left(\frac{H}{L}\right)^2\right]; \qquad i=1,2,3 \tag{45}$$

The constant K is defined in Eq. (35).

The present theory solutions correspond to setting all the tracer constants to unity; they are exact for the end conditions imposed. The designation C1, C2, and C3 has been chosen to correspond to the order of increasing stiffness of the end restraint. Differences in response due to the definition of clamping imposed are minor in this case.

If K_i and K are set to zero in the preceding equations, the Bernoulli-Euler results are obtained. This approximation overestimates stiffness. The Timoshenko theory results ($\alpha_s=1$ and $\alpha_a=\alpha_n=0$), interestingly enough, overestimate stiffness in this case.

Bernoulli-Euler theory tends to always overestimate stiffness. Timoshenko theory, however, in light of the results presented here, may either provide an overestimate or underestimate of stiffness, depending upon the problem under consideration.

A related problem is a cantilever beam subjected to a linearly varying distributed load that varies from zero at the free end to q at the fixed end. The exact solution is given in Ref. 4. For an isotropic beam with C1 restraint at the fixed end, the tip deflection is

$$\delta = \frac{qL^4}{30EI}\left\{1+\frac{H^2}{4L^2}\left[2\alpha_a+5\alpha_n+5\alpha_s(1+\nu)\right] - \underline{\frac{\alpha_s}{4}(1+\nu)\left(\frac{H}{L}\right)^4}\right\} \tag{46}$$

If the present theory is used, the underlined term is not obtained. For $L/H>2$, this term is negligible. For practical purposes, therefore, the present theory results are indistinguishable from the exact ones.

Clamped Beam

Unlike the previous examples, the clamped beam is statically indeterminate. Three solutions corresponding to the three clamping models have been determined. They may be expressed in a common form. It is convenient to place the origin of coordinates at midspan and use the semilength ℓ. The bending moment distribution is

$$M=(q/2)(\ell^2-x^2)-M_0 \tag{47}$$

M_0 is the end fixing moment, which is positive if it tends to reduce the end rotation due to the uniform load.

The expression for the lateral axis deflection is

$$W=\frac{1}{E_{11}I}\left[\frac{q}{4}\left(\frac{5}{6}\ell^4-\ell^2x^2+\frac{x^4}{6}\right)+\left(\frac{KH^2q}{8}-\frac{M_0}{2}\right)(\ell^2-x^2)\right] \tag{48}$$

The redundant end fixing moment is different for each clamping model. It can be written in the common form

$$M_0=\frac{qL^2}{12}\left[1+3\left(\frac{H}{L}\right)^2(K-K_i)\right]; \qquad i=1,2,3 \tag{49}$$

The occurrence of different end moment values is due to the statically indeterminate nature of the structure. The present theory yields the exact solutions to this problem for each form of clamping.

A central moment ratio as a function of beam slenderness is plotted in Figs. 3 and 4 for isotropic and orthotropic materials, respectively. The notation B-E refers to the value from Bernoulli-Euler theory. Bernoulli-Euler and Timoshenko theories give identical predictions. The present theory, however, which agrees with the exact solutions, predicts behavior that differs for each clamping model. Departures from classical theory are much greater for beams made of the typical orthotropic material. This is because the transverse shear stiffness to extensional stiffness ratio is low for this material, which accentuates the nonclassical effects.

It is apparent that predictions for shorter beams, the class of structures for which improved predictions are the most needed, are sensitive to the precise conditions used to model restraint. Consequently, care must be devoted to matching mathematical descriptions of boundary restraint with practical end restraint achieved in tests or structural assemblies. The clamped end condition is an extreme example as cross section warping cannot be fully eliminated by elementary means.

Concluding Remarks

A systematic analysis employing tracer constants has been presented which adds quantitative proof to the earlier statements of Goodier. There are two additional effects, transverse normal strain and nonclassical axial stress, in addition to transverse shear strain, which must be included in a refined bending theory. New equations for planar bending are developed which 1) include all the essential physical effects, 2) provide predictions which are exact or indistinguishable from exact solutions for a variety of examples, 3) provide information which is not discernible by other engineering theories, and 4) are as simple to apply as static Timoshenko-type theory.

Further consideration must be given to the modeling of boundary conditions which restrain local cross section warping. This will be the subject of a forthcoming paper.

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